

Chapter 22  
The Hamiltonian and Lagrangian densities  
—  
from my book:  
Understanding Relativistic Quantum Field Theory

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## **Chapter 22**

# **The Hamiltonian and Lagrangian densities**

## 22.1 The relativistic Hamiltonian and Lagrangian

The Hamiltonian and Lagrangian which are rather abstract constructions in classical mechanics get a very simple interpretation in relativistic quantum mechanics. Both are proportional to the number of phase changes per unit of time. The Hamiltonian runs over the time axis while the Lagrangian runs over the trajectory of the moving particle, the  $t'$ -axis.

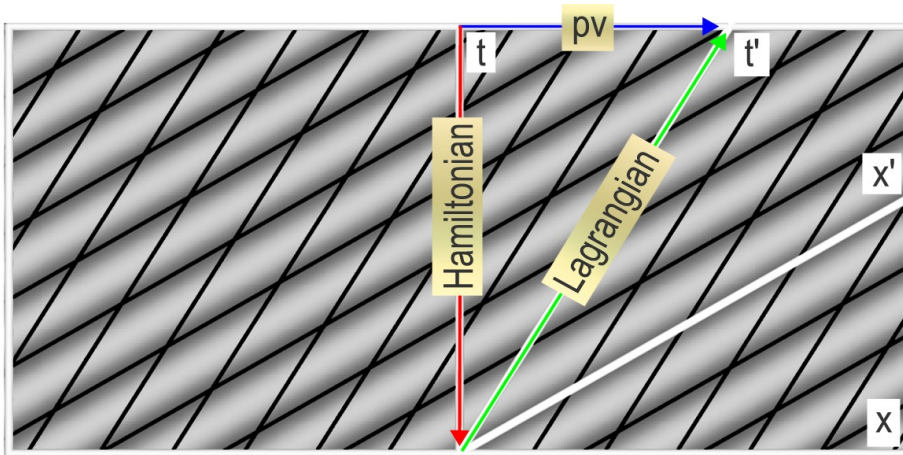


Figure 22.1: The Hamiltonian and Lagrangian

Figure 22.1 shows the relativistic de Broglie wave in a Minkowski diagram. The triangle represents the relation between the Lagrangian and the Hamiltonian, which holds in both relativistic and non-relativistic physics.

$$L = pv - H \quad (22.1)$$

The Hamiltonian counts the phase-changes per unit of time on the vertical axis while the term  $pv$  counts the phase-changes per unit on the horizontal axis:  $v$  is the distance traveled per unit of time while  $p$  is proportional with the phase-changes per unit of distance, hence the term  $pv$ . We can now understand the classical relation. (with  $\dot{q} = \dot{x} = v$ )

$$\frac{\partial L}{\partial \dot{q}} = p \quad (22.2)$$

For the free classical relativistic particle we have for the Hamiltonian (Energy) and the  $pv$  term.

$$H = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad pv = \frac{mv^2}{\sqrt{1 - \frac{v^2}{c^2}}} \quad (22.3)$$

Calculating the Lagrangian we see that the Hamiltonian is proportional to  $\gamma$  while the Lagrangian is proportional to  $1/\gamma$ .

$$L = -(H - pv) = -\frac{c^2 - v^2}{\sqrt{1 - \frac{v^2}{c^2}}} m = -\sqrt{1 - \frac{v^2}{c^2}} mc^2 \quad (22.4)$$

This is what we expect from time dilation. The moving particle has less clock-ticks by a factor  $\gamma$  due to the time dilation, We now check that.

$$\frac{\partial L}{\partial \dot{q}} = \frac{\partial}{\partial v} \left\{ -\sqrt{1 - \frac{v^2}{c^2}} mc^2 \right\} = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} = p \quad (22.5)$$

For sofar we have not yet discussed the potential energy. To obtain the equation of motion of a relativistic particle in a potential field we have to add the potential energy term  $V(q)$ . In the non-relativistic case we have.

$$H = T(\dot{q}) + V(q), \quad L = T(\dot{q}) - V(q) \quad (22.6)$$

Where  $T(\dot{q}) = \frac{1}{2}mv^2$  is the kinetic energy. The relativistic Hamiltonian and Lagrangian we have discussed however also include the restmass energy. The restmass energy can be considered as being part of the potential energy. The kinetic part  $T$  in the relativistic case can be obtained as follows.

$$\begin{aligned} H + L &= 2T = pv \quad \Rightarrow \\ T &= \frac{1}{2}pv = \frac{1}{2} \frac{mv^2}{\sqrt{1 - \frac{v^2}{c^2}}} \approx \frac{1}{2}mv^2 \quad (\text{for } v \ll c) \end{aligned} \quad (22.7)$$

Using the term  $L = \frac{1}{2}mv^2$  in eq. (22.5) gives us  $p = mv$  for the non-relativistic momentum

## 22.2 Principle of least action / least proper time

To obtain the relativistic equation of motion of a particle in a potential field we use Lagrange equation of motion derived from Hamilton's variational principle of Least Action. Relativistic quantum mechanics gives us the deeper perspective of this principle which now becomes the principle of least phase-change and thus the least proper time: The relativistic particle follows the path which will bring it there in the shortest proper time.

We will briefly recall the derivation here, it is one of the most fundamental principles of physics. Somewhat abstract we can write.

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}) dt = 0 \quad (22.8)$$

Where  $\delta S$  is the variation of the action, the variation of the proper time in quantum mechanics, is zero, meaning that we have a minimum or maximum just like the a zero first order derivative of a function indicates a local minimum or maximum. The particle traverses a path between two fixed points  $x_1$  and  $x_2$  between  $t_1$  and  $t_2$  which are also fixed. We assume that the Lagrangian only depends on the position  $q$  of the particle and its velocity  $\dot{q}$ . Now we allow small arbitrary variations in  $q$  and  $\dot{q}$  along the path.

$$\delta S = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \delta \dot{q} \right) dt = 0 \quad (22.9)$$

As long as these variations are small we can determine the change in  $L$  with the help of the first order derivatives. The last term above is also the last term in the expression below according to the product rule for differentiation.

$$\frac{d}{dt} \left\{ \frac{\partial L}{\partial \dot{q}} \delta q \right\} = \frac{d}{dt} \left\{ \frac{\partial L}{\partial \dot{q}} \right\} \delta q + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt} \left\{ \delta q \right\} \quad (22.10)$$

We can instantly integrate the left hand term which gives us the following.

$$\delta S = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q} - \frac{d}{dt} \left\{ \frac{\partial L}{\partial \dot{q}} \right\} \right) \delta q dt + \left. \frac{\partial L}{\partial \dot{q}} \delta q \right|_{t_1}^{t_2} = 0 \quad (22.11)$$

The integrated term at the end vanishes since  $\delta q$  is defined as zero at the end-points. Since  $\delta q$  is arbitrary along the rest of the path we have to set the term between brackets to zero. This then gives us the Euler-Lagrange equation.

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \quad (22.12)$$

It defines a local minimum at each point along the path. We can use the metaphor of a ball rolling at the bottom of a valley. Inserting the Lagrangian of the classical relativistic particle.

$$L = -\sqrt{1 - \frac{v^2}{c^2}} mc^2 - V(q) \quad (22.13)$$

Gives us the *equation of motion* of the relativistic particle in a potential field.

$$\frac{dp}{dt} = -\frac{dV}{dq}, \quad \text{with} \quad p = \frac{mv}{\sqrt{1 - v^2/c^2}} \quad (22.14)$$

A gradient of the potential field causes a change in the relativistic momentum: If we use the non-relativistic Lagrangian with,

$$L \approx \frac{1}{2}mv^2 - V(q) \quad (22.15)$$

then we get the non-relativistic equation of motion.

$$ma = -\frac{dV}{dq} \quad (22.16)$$

As long as the derivation is correct for the relativistic particle, then we can be assured that it is valid for the non-relativistic limit as well.



## 22.3 The Hamiltonian and Lagrangian density

We can define the Hamiltonian and Lagrangian *density* for any extended object, being either classical or a quantum field, as.

$$H = \int \mathcal{H} dx^3, \quad L = \int \mathcal{L} dx^3 \quad (22.17)$$

Let us see how these quantities transform under Lorentz transformation. We did see in (22.3) and (22.4) that the integrated quantities, the Hamiltonian  $H$ , the Lagrangian  $L$  and the term  $pv$  transform like.

$$\begin{aligned} H & \text{ transforms as } \gamma \\ pv & \text{ transforms as } \beta^2 \gamma \\ L & \text{ transforms as } 1/\gamma \end{aligned} \quad (22.18)$$

The volume of a wave-functions transforms like  $1/\gamma$  due to Lorentz contraction. So, the densities become higher by a factor  $\gamma$ , hence the density  $\mathcal{H}$ , the Lagrangian density  $\mathcal{L}$  and the density of the  $pv$  term transform like.

$$\begin{aligned} \mathcal{H} & \text{ transforms as } \gamma^2 \\ pv & \text{ transforms as } \beta^2 \gamma^2 \\ \mathcal{L} & \text{ transforms as } "1" \end{aligned} \quad (22.19)$$

We see that the Lagrangian density is the same in all reference frames. It is a Lorentz scalar. This makes the Lagrangian density a fundamental quantity in quantum field theory. The Standard Model of physics is based on the Lagrangian density which in quantum physics is generally called just the Lagrangian, without the density.

The "equations of motion" are based on the derivatives of the Lagrangian density which is a Lorentz scalar. Done in the right way assures that the whole Standard Model of physics transforms in the right way, that is, the laws of physics are the same in every reference frame.

The triangular equation  $H - pv = -L$  basically counts phase change "clock-pulses" on the  $t$ ,  $x$  and  $t'$ -axis. The corresponding relation of the densities transforms like the basic energy/momentum relation.

$$\mathcal{H} - pv = -\mathcal{L} \quad \text{transforms as} \quad E^2 - p^2 = m^2 \quad (22.20)$$

## 22.4 The Euler-Lagrange equation for Fields

The Euler-Lagrange equation for fields operates on a lagrangian which depends only on the (generalized) coordinate  $q$  and velocity  $\dot{q}$  of the particle. It is valid for relativistic particles even though it was developed by Leonhard Euler and Joseph-Louis Lagrange in the 1750's.

$$L \equiv L(q, \dot{q}) \quad (22.21)$$

In quantum field theory we do not have a quantity like  $q$  explicitly available. We work with a field  $\psi(t, r)$  instead. We assume that the Lagrangian density only depends on  $\psi(t, r)$  and its first order derivatives.

$$\mathcal{L} \equiv \mathcal{L}(\psi, \partial_t\psi, \partial_x\psi, \partial_y\psi, \partial_z\psi) \quad (22.22)$$

We will avoid in this book the widespread custom to present the correct relativistic Lagrangian density for the (scalar) Klein Gordon field and then justify it by making substitutions like.

$$q \leftarrow \psi, \quad \frac{1}{2}mv^2 \leftarrow \frac{1}{2}\dot{\psi}^2 \quad (22.23)$$

The term  $\frac{1}{2}mv^2$  should worry the reader. Indeed the Hamiltonian density subsequently derived does transform in the wrong way and its integral over space does not correspond with the Hamiltonian of the classical relativistic particle.

The origin of these substitutions can be understood by looking at our initial mechanical spring-mass model of the Klein Gordon equation shown in figure ???. In this model  $\psi$  is a displacement which could be associated with  $q$ . These substitutions however are too naive, worse, they lead to violations of special relativity.

Rather than trying to make associations between classical and field terms we want to stress the fact that the Euler-Lagrange mechanism to derive the equations of motion is an entirely *mathematical* mechanism to find a minimum/maximum. It does not matter by what quantities the Lagrangian density is expressed as long as they express the right one.

The Euler-Lagrange equation for quantum fields goes well beyond the scalar Klein Gordon field. It holds for all quantum fields. The

derivation is the same as the derivation for the classical one. We will follow the same four steps as in equations (22.9) through (22.13) for the classical wave equation. The variation of the action is symbolized by.

$$\delta S = \delta \int_{t_1}^{t_2} dt \int_{-\infty}^{+\infty} \mathcal{L}(\psi, \partial_t \psi, \partial_x \psi, \partial_y \psi, \partial_z \psi) dx^3 = 0 \quad (22.24)$$

More concretely we can write it expressed in variations of the field  $\psi$  and its (four) derivatives.

$$\delta S = \int_{t_1}^{t_2} \left( \frac{\partial \mathcal{L}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \delta \partial_\mu \psi \right) dx^4 = 0 \quad (22.25)$$

The last term above is equal to the last term in the equation below which just expresses the product rule for differentiation. In fact there are four terms in total, one for each derivative.

$$\frac{\partial}{\partial \mu} \left\{ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \delta \psi \right\} = \frac{\partial}{\partial \mu} \left\{ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \right\} \delta \psi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \frac{\partial}{\partial \mu} \left\{ \delta \psi \right\} \quad (22.26)$$

The left hand term above (representing four terms) can be directly integrated over one axis, each of the four along its own axis. The result of these integrations is zero since the variations at the end points are defined as zero. So, we can omit the integration over the other three axis and continue with the remaining terms which are all proportional to the variation of  $\psi$  itself.

$$\delta S = \int_{t_1}^{t_2} \left( \frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial \mu} \left\{ \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \right\} \right) \delta \psi dx^4 = 0 \quad (22.27)$$

Since  $\delta \psi$  is totally arbitrary we conclude that the equation between brackets has to hold at each point (This means we have nothing to do anymore with where the borders are in which reference frame). We have obtained the Euler Lagrange equation for the relativistic Lagrangian density:

$$\frac{\partial}{\partial \mu} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \quad (22.28)$$

## 22.5 Lagrangian of the scalar Klein Gordon field

The form of the Hamiltonian and Lagrangian densities of the Klein Gordon field are determined by the fact the Klein Gordon field is a scalar field. This means that the values of the field  $\psi$  are Lorentz invariant. They are the same in any reference frame.

If we perform a Lorentz transform on the field like  $\psi'(x^\mu) = \Lambda\psi(x^\mu)$  then it suffices to transform the coordinates  $x^\mu$  to obtain  $\psi'(x^\mu)$

$$\psi'(x^\mu) = \Lambda\psi(x^\mu) = \psi(\Lambda^{-1}x^\mu) \quad (22.29)$$

We did see that the Hamiltonian density transforms as  $E^2$ , while the kinetic (density) term  $\mathcal{T}$  transforms like  $p^2$ , see (22.20). Since the scalar field  $\psi$  itself doesn't transform we might expect differential operators which obtain  $E^2$  and  $p^2$  from the quantum field instead. We will see that this is indeed the case.

In the classical non relativistic theory the Lagrangian is given by  $L = T - V$ . In case of a relativistic free particle there is no potential energy, however there is the energy corresponding with the mass of the particle. This 'self-energy' term, which plays a somewhat similar role as the potential energy, is absent in the non-relativistic theory.

We did see that the kinetic term  $T$  becomes  $\frac{1}{2}pv$  in the relativistic case. We want to write the relativistic case in a form of  $L = T - W$ , where  $W$  relates to the mass-energy. Using the generally valid  $L = pv - H$  we can rewrite  $L$  like:

$$L = -\frac{1}{2}H + \frac{1}{2}pv + \frac{1}{2}L \quad (22.30)$$

For a classical relativistic particle we should substitute the right hand terms as follows (see section 22.1)

$$L = -\frac{1}{2}\gamma mc^2 + \frac{1}{2}\gamma mv^2 - \frac{1}{2}\gamma^{-1}mc^2 \quad (22.31)$$

Going from the Lagrangian to the Lagrangian density we have to multiply the right hand terms with an extra factor  $\gamma$  to compensate for Lorentz

contraction which confines the wave-function into a smaller volume, hence the density goes up by a factor  $\gamma$ . At this point we will also multiply all right hand terms with the constant  $mc^2$  for convenience. We will see why. So, all the right hand terms are multiplied by  $\gamma mc^2$  when going from the Lagrangian  $L$  to the Lagrangian density  $\mathcal{L}$ .

$$\mathcal{L} = -\frac{1}{2} [\gamma mc^2]^2 + \frac{1}{2} [\gamma mv]^2 c^2 - \frac{1}{2} [mc^2]^2 \quad (22.32)$$

The terms between square brackets we recognize as  $E$ ,  $p$  and the rest-mass energy. We can now make the step from the classical relativistic particle to the Klein Gordon field theory:

$$\mathcal{L} = \frac{1}{2} \left[ \hbar \frac{\partial \psi}{\partial t} \right]^2 - \frac{1}{2} \left[ \hbar \frac{\partial \psi}{\partial x^i} \right]^2 c^2 - \frac{1}{2} [mc^2 \psi]^2 \quad (22.33)$$

Setting ( $c = \hbar = 1$ ) gives us the familiar form of the Lagrangian density of the Klein Gordon field.

$$\mathcal{L} = \frac{1}{2} \dot{\psi}^2 - \frac{1}{2} \nabla \psi \cdot \nabla \psi - \frac{1}{2} m^2 \psi^2 \quad (22.34)$$

Note that the middle term at the right hand side corresponds with the classical kinetic term  $T$  which becomes the kinetic energy in the non relativistic theory of the classical particle. We have derived the Lagrangian density for the scalar field. By assuming that  $\psi$  is a Lorentz scalar we obtained the above Lagrangian density. We now use the Euler Lagrange equation.

$$\frac{\partial}{\partial \mu} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \quad (22.35)$$

In order to obtain the equation of motion. What we get is the Klein Gordon equation.

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x_i^2} + m^2 \psi = 0 \quad (22.36)$$

The scalar quantum field representation derived directly from the classical relativistic particle Lagrangian.

## 22.6 Hamiltonian of the scalar Klein Gordon field

For the Hamiltonian density we go back to the expression for the classical relativistic particle since we want both Hamiltonians to correspond. Using equation (22.32) and  $H = 2T - L$  we get for the Hamiltonian density of the classical particle.

$$\mathcal{H} = \frac{1}{2} [\gamma mc^2]^2 + \frac{1}{2} [\gamma mv]^2 c^2 + \frac{1}{2} [mc^2]^2 \quad (22.37)$$

This corresponds with the following Hamiltonian density for the Klein Gordon equation.

$$\mathcal{H} = -\frac{1}{2}\dot{\psi}^2 - \frac{1}{2}\nabla\psi \cdot \nabla\psi + \frac{1}{2}m^2\psi^2 \quad (22.38)$$

With ( $c = \hbar = 1$ ). We can see how this Hamiltonian density transforms by applying it to a plane-wave of the form  $\exp(-iEt + ipx)$ . We get:

$$\mathcal{H} \propto E^2 + p^2 + m^2 \propto \gamma^2 + \beta^2\gamma^2 + 1 \propto \gamma^2 \quad (22.39)$$

Thus: the Hamiltonian *density* transforms like  $\gamma^2$  where one factor  $\gamma$  stems from the Hamiltonian being the 0'th component of the 4-momentum and the second factor  $\gamma$  comes from the Lorentz contraction of the volume which confines the field.

This Hamiltonian density corresponds with the classical particle but differs in signs with the second quantization related Hamiltonian density<sup>1</sup>

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<sup>1</sup>The Hamiltonian related with the second quantization of the Klein Gordon field is given by  $\mathcal{H} = \frac{1}{2}\dot{\psi}^2 + \frac{1}{2}(\nabla\psi)^2 + \frac{1}{2}m^2\psi^2$ . The reason of the difference in signs is that the term  $\dot{\psi}$  is considered as the momentum in an "internal" or "unspecified" space. The term  $\frac{1}{2}\dot{\psi}^2$  is then considered to be the  $T$  in  $H = 2T - L$ . The general problem in applying second order quantization in the relativistic theory is that the mix of "internal" or "unspecified" space and the usual space-time coordinates doesn't lead to the correct Lorentz transformation. A second problem is that  $\frac{1}{2}\dot{\psi}^2$  corresponds to the non-relativistic expression  $\frac{1}{2}mv^2$  instead of the relativistic version  $\frac{1}{2}pv$

## 22.7 The complex Klein Gordon field

The complex Klein Gordon equation comes in when we need to describe both particles and anti-particles.  $\psi$  becomes a field with two components  $\psi_1$  and  $\psi_2$ .

$$\psi = \psi_1 + i\psi_2 \quad (22.40)$$

How can we interpret these components. One possibility is to take one component as a position (in some "internal" or "unspecified" space and the other component as the momentum. The two would be  $90^\circ$  out of phase in an oscillatory motion which we could associate with the particle's frequency  $\exp(-iEt/\hbar)$

Another way is to interpret both as coordinates on a plane of rotation. The expression  $\exp(-iEt/\hbar)$  would then correspond with a circular motion<sup>2</sup>.

The Lagrangian for the complex field must contain both components and the Euler-Lagrange equation must lead to the usual Klein Gordon equation. Say we use the Lagrangian of the real Klein Gordon equation,

$$\mathcal{L} = \frac{1}{2}\dot{\psi}^2 - \frac{1}{2}\nabla\psi \cdot \nabla\psi - \frac{1}{2}m^2\psi^2 \quad (22.41)$$

and simply use  $\psi$  as a complex variable. Evaluating this with a plane wave solution however doesn't produce a real value.  $\mathcal{L}$  contains the local phase of the wave-function, its not a real value.

$$\mathcal{L} = \left( -\frac{1}{2}E^2 + \frac{1}{2}p^2 - \frac{1}{2}m^2 \right) \psi^2 = -m^2\psi^2 \quad (22.42)$$

The result we want is  $-m^2|\psi|^2$  or  $-m^2\psi^*\psi$  which more explicitly expressed in its individual components is  $-m^2(\psi_1^2 + \psi_2^2)$  without the imaginary valued cross-term  $-2im^2\psi_1\psi_2$  included in  $m^2\psi^2$ . We get the Lagrangian density we want by simply adding the two Lagrangians of the individual components together. The same can be done for the Hamiltonian.

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<sup>2</sup>Both interpretations lead to a specific direction in space, either the direction of oscillation or the spin pointer. This is an issue for a scalar theory such as the one represented by the scalar Klein Gordon equation, which is not supposed to have any special direction in space

**Lagrangian density for the complex scalar field  $\psi_1 + i\psi_2$**

$$\mathcal{L} = \left\{ \begin{array}{l} + \left( \frac{1}{2} \dot{\psi}_1^2 - \frac{1}{2} \nabla \psi_1 \cdot \nabla \psi_1 - \frac{1}{2} m^2 \psi_1^2 \right) \\ + \left( \frac{1}{2} \dot{\psi}_2^2 - \frac{1}{2} \nabla \psi_2 \cdot \nabla \psi_2 - \frac{1}{2} m^2 \psi_2^2 \right) \end{array} \right\} \quad (22.43)$$

$$\mathcal{L} = \frac{1}{2} \left( -\dot{\psi}^* \dot{\psi} + \nabla \psi^* \cdot \nabla \psi - m^2 \psi^* \psi \right) \quad (22.44)$$

**Hamiltonian density for the complex scalar field  $\psi_1 + i\psi_2$**

$$\mathcal{H} = \left\{ \begin{array}{l} + \left( -\frac{1}{2} \dot{\psi}_1^2 - \frac{1}{2} \nabla \psi_1 \cdot \nabla \psi_1 + \frac{1}{2} m^2 \psi_1^2 \right) \\ + \left( -\frac{1}{2} \dot{\psi}_2^2 - \frac{1}{2} \nabla \psi_2 \cdot \nabla \psi_2 + \frac{1}{2} m^2 \psi_2^2 \right) \end{array} \right\} \quad (22.45)$$

$$\mathcal{H}(x) = \frac{1}{2} \left( \dot{\psi}^* \dot{\psi} + \nabla \psi^* \cdot \nabla \psi + m^2 \psi^* \psi \right) \quad (22.46)$$

Some care is required with the signs here since  $\dot{\psi}^* \dot{\psi} = -(\dot{\psi}_1^2 + \dot{\psi}_2^2)$ . Equation (22.44) is often found with reversed signs. One can easily check the required signs by inserting a plane-wave eigenfunction into the Lagrangian density:  $\psi = \psi_1 + i\psi_2 = \cos(-Et + px) + i \sin(-Et + px)$ , One should get.

$$\mathcal{L} = \frac{1}{2} \left( -E^2 + p^2 - m^2 \right) \psi^* \psi = -m^2 \psi^* \psi \quad (22.47)$$

The equation of motion can be derived in a mathematically proper way<sup>3</sup> by applying the Euler-Lagrange equation on (22.43), taking the derivatives in the fields  $\psi_1$ ,  $\psi_2$  and their derivatives, and then defining the combined field again as  $\psi = \psi_1 + i\psi_2$ . The result is the familiar Klein Gordon equation.

$$\frac{\partial^2 \psi}{\partial t^2} - \frac{\partial^2 \psi}{\partial x_i^2} = m^2 \psi \quad (22.48)$$

<sup>3</sup>Despite what is often seen, the Euler Lagrange equation can not be applied directly on expressions containing terms like  $\psi^* \psi$ . A derivative like  $\partial \psi^* / \partial \psi$  is not zero but undetermined since it violates the Cauchy-Riemann equations for complex differentiability: The derivative is not independent of the direction of  $\partial \psi$  in the complex plane.



## 22.8 Expressing the Lagrangian in a 4d environment

The reader may have noticed that there seems to be an ambiguity in how we define the Lagrangian density. The expression for the Lagrangian density we obtained is.

$$\mathcal{L} = \frac{1}{2} \left( -\dot{\psi}^2 + (\nabla\psi)^2 - m^2\psi^2 \right) \quad (22.49)$$

While we in fact could also have written.

$$\mathcal{L} = -m^2\psi^2 \quad \text{or} \quad \mathcal{L} = -\dot{\psi}^2 + (\nabla\psi)^2 \quad (22.50)$$

Both expressions lead to the amount of phase change per unit of time per unit of volume over the trajectory of the particle, at least for plane-waves. Instead we end up with a linear combination of the latter two expressions. Why? For the Hamiltonian it seems that we could equally well write.

$$\mathcal{H} = \dot{\psi}^2 \quad \text{or} \quad \mathcal{H} = (\nabla\psi)^2 + m^2\psi^2 \quad (22.51)$$

To get the amount of phase change per unit of time per unit of volume over the time-axis. Instead we ended up with.

$$\mathcal{H}(x) = \frac{1}{2} \left( \dot{\psi}^2 + (\nabla\psi)^2 + m^2\psi^2 \right) \quad (22.52)$$

One important argument is that we do not only want to know the Lagrangian or Hamiltonian density in one particular reference frame, but we want to know these quantities in all reference frames. Knowing a phase change rate in one particular space-time direction doesn't say anything about the other directions. We need sufficient information to be able to transform the Lagrangian and Hamiltonian density into any reference frame.

One might suspect that Nature itself also needs such a definition in 4d space-time, and that therefor, even though the expressions seem to be ambiguous from a single reference point of view, they actually do represent the physics as required in a 4d-dimensional world obeying the rules of special relativity.

## 22.9 Electromagnetic and Proca Lagrangian

In correspondence with the Lagrangian densities discussed sofar we might expect the Lagrangian density for the electromagnetic four-vector to be expressed by the following.

$$\mathcal{L}_p = \left\{ \begin{array}{l} + \left( \frac{1}{2} \dot{A}_0^2 - \frac{1}{2} \nabla A_0 \cdot \nabla A_0 - \frac{1}{2} m^2 A_0^2 \right) \\ - \left( \frac{1}{2} \dot{A}_x^2 - \frac{1}{2} \nabla A_x \cdot \nabla A_x - \frac{1}{2} m^2 A_x^2 \right) \\ - \left( \frac{1}{2} \dot{A}_y^2 - \frac{1}{2} \nabla A_y \cdot \nabla A_y - \frac{1}{2} m^2 A_y^2 \right) \\ - \left( \frac{1}{2} \dot{A}_z^2 - \frac{1}{2} \nabla A_z \cdot \nabla A_z - \frac{1}{2} m^2 A_z^2 \right) \end{array} \right. \quad (22.53)$$

Where all four components of  $A^\nu$  have independently the form of the (classical) Lagrangian field density. The requirement that the total Lagrangian density transforms like a Lorentz scalar imposes the (+,-,-,-) metric on the time/space components, the signs in the first column of (22.53). These expressions can be written more compact as:

$$\mathcal{L}_p = \frac{1}{2} \partial^\mu A^\nu \partial_\mu A_\nu - \frac{1}{2} m^2 c^2 A^\nu A_\nu \quad (22.54)$$

If the mass  $m$  is not zero then we call the field a *Proca field*. This expression is however not complete. We have to replace the derivatives of  $A^\mu$  in the following sense. (We'll discuss the reason for this in a minute)

$$\partial^\mu A^\nu \implies \partial^\mu A^\nu - \partial^\nu A^\mu, \quad \partial_\mu A_\nu \implies \partial_\mu A_\nu - \partial_\nu A_\mu \quad (22.55)$$

So the Lagrangian density becomes. (in the massless case)

$$\mathcal{L} = \frac{1}{2} (\partial^\mu A^\nu - \partial^\nu A^\mu) (\partial_\mu A_\nu - \partial_\nu A_\mu) = F^{\mu\nu} F_{\mu\nu} \quad (22.56)$$

Which we can write (using the normalization in SI) as.

$$\mathcal{L} = \frac{1}{4\mu} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}(\text{BH} - \text{DE}) \quad (22.57)$$

This Lagrangian density is zero in case of electromagnetic radiation where the relation  $|E| = |cB|$  holds always. This shows us that the invariant photon mass is zero. (The partial Lagrangian density (22.54) is zero as well in this case)

Now, why do we need to subtract these extra terms? Well typically the energy-momentum of the electromagnetic field is derived by calculating how the field acts on charges on a capacitor and currents in an inductor, so charge is involved in one way or the other and charge is represented by an U(1) symmetry:  $\exp(i\phi)$ .

The terms we need to subtract induce an equal and indistinguishable U(1) phase as the regular Lagrangian components and therefor need to be taken care of. The total induced phase  $\phi$  on a charged scalar field  $\psi$  by the four-vector  $A^\mu$  is defined in the following way.

$$\psi = \exp \left[ -\frac{i}{\hbar} \int (p^o + eA^o) dx^o + \frac{i}{\hbar} \sum_{i=1}^3 \int (p^i + eA^i) dx^i \right] \quad (22.58)$$

Where  $p$  is the inertial momentum defined by the invariant mass. We can split of the factor dependent on  $A^\mu$  as follows  $\psi = \psi^p \psi^A$ . Since  $\psi$  is a scalar we can write  $\partial^\mu \partial^\nu \psi = \partial^\nu \partial^\mu \psi$ , the order of the differential operators does not matter.

The combination  $(p_\mu + eA_\mu)$  must be curl free in any of the six 2D planes of 4D space-time otherwise  $\psi$  otherwise the expression within the square brackets can not be a single valued scalar function.

$$\oint (p_\mu + eA_\mu) ds = 0 \quad (22.59)$$

The individual terms  $p_\mu$  and  $eA_\mu$  can have curl, so any curl in  $eA_\mu$  must be canceled by an opposite curl in  $p_\mu$ . This relationship gives rise to the electromagnetic Lorentz Force. The expression  $i\hbar \partial^\mu \partial^\nu \psi^p$  yields the

changes of the momenta  $p^\nu$  in the directions  $x^\mu$ . All of these terms end up in the expression which describes the change of momentum of a particle moving at some speed  $v$  since.

$$\frac{dp^\mu}{dt} = \frac{\partial p^\mu}{\partial t} + \frac{\partial p^\mu}{\partial x} v_x + \frac{\partial p^\mu}{\partial y} v_y + \frac{\partial p^\mu}{\partial z} v_z \quad (22.60)$$

Where the right hand side is just the mathematical expansion of the left hand side ( $v_x = \partial x / \partial t$ , et-cetera ) It is no surprise that the expression  $\partial^\mu \partial^\nu \psi^A$  gives rise to the terms which end up in the electromagnetic field tensor  $F^{\mu\nu}$  which yields the changing momentum.

$$\frac{dp^\mu}{dt} = e \left( \partial^\mu A^\nu - \partial^\nu A^\mu \right) v^\nu = e F^{\mu\nu} v^\nu \quad (22.61)$$

Where the term between brackets represents the curl of  $A^\mu$ . Since the order of differentiation is irrelevant we can deduce.

$$\partial^\mu \partial^\nu \psi = \partial^\nu \partial^\mu \psi \quad \implies \quad \partial^\mu (p^\nu + eA^\nu) = \partial^\nu (p^\mu + eA^\mu) \quad (22.62)$$

The righthand side can be reordered as.

$$\partial^\mu p^\nu - \partial^\nu p^\mu = -e (\partial^\mu A^\nu - \partial^\nu A^\mu) \quad (22.63)$$

This expression simplifies for a particle with  $\vec{v} = 0$ , and thus  $\vec{p} = 0$  (at every point in space) to.

$$\frac{\partial p^i}{\partial x^0} = -e \frac{\partial A^i}{\partial x^0} - e \frac{\partial^\nu A^0}{\partial x^i} = e \mathbf{E} \quad (22.64)$$

Which is just the electric part of the Lorentz force. A non-zero velocity  $\vec{v}$ , constant over space, gives rise to the full Lorentz force including the magnetic terms.

We see that the assumption that somehow charge is involved in the Langrangian leads to extra terms which are indistinguishable from the basic

terms because of the U(1) symmetry. Certainly a term like  $\frac{1}{2}\mathbf{D}\mathbf{E}$  suggests likewise. It is the energy needed for the vacuum displacement current  $\mathbf{D}$  to move through the electric field  $\mathbf{E}$  which builds up in the process as a result.

## 22.10 The electromagnetic equation of motion

We can derive the equations of motion with the use of the Euler Lagrange equation starting from the Lagrangian density.

$$\mathcal{L} = \frac{1}{4\mu_o} \left( \partial_\mu A_\nu - \partial_\nu A_\mu \right) \left( \partial^\mu A^\nu - \partial^\nu A^\mu \right) = \frac{1}{2\mu_o} \left( B^2 - \frac{1}{c^2} E^2 \right) \quad (22.65)$$

Multiplying the terms and realizing that  $\mu$  and  $\nu$  are dummy variables to add all terms together to a single scalar result, we can rewrite this in the form which is generally used to derive the equations of motion.

$$\mathcal{L} = \frac{1}{2\mu_o} \left( \partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu \right) \quad (22.66)$$

The first of the two terms is identical to the initial Lagrangian density (22.53) while the second term corresponds to the extra terms. We recall the Euler Lagrange equation.

$$\frac{\partial}{\partial \mu} \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu A^\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A^\nu} = 0 \quad (22.67)$$

Applying it gives us the equation of motion.

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = 0 \quad (22.68)$$

The term  $\partial_\mu A^\mu$  is zero if  $A^\mu$  is a conserved current. We see that the equation of motion is not changed by the extra terms if this is the case. In case of a vacuum without net charge-current density we obtain.

$$\partial_\mu \partial^\mu A^\nu = 0 \quad (22.69)$$

## 22.11 The electromagnetic interaction Lagrangian

The electromagnetic interaction Lagrangian density represents the extra phase change rates induced by the four-potential  $A^\mu$  on a charged (Klein Gordon) field  $\psi$ . It is given by.

$$\mathcal{L}_{int} = j_\nu A^\nu \quad (22.70)$$

Which transforms as it should like a Lorentz scalar. This gives us for the total equation of motion.

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \mu_o j^\nu \quad (22.71)$$

Which becomes under the assumption that  $A^\mu$  is a conserved current (Lorentz gauge condition).

$$\partial_\mu \partial^\mu A^\nu = \mu_o j^\nu \quad (22.72)$$

Here we recover the classical the classical wave equation of  $A^\mu$  with the charge-current density  $j^\mu$  as its source. We can write the equation of motion as.

$$\partial_\mu \left( \partial^\mu A^\nu - \partial^\nu A^\mu \right) = \mu_o j^\nu \quad (22.73)$$

This allows us to express it with the use of  $F^{\mu\nu}$  instead of  $A^\nu$ .

$$\partial_\mu F^{\mu\nu} = \mu_o j^\nu \quad (22.74)$$

Which is a compact way of writing of the inhomogeneous Maxwell equations, which we have hereby derived from the Lagrangian density.

$$\nabla \cdot \mathbf{E} = c\mu_o j^o = \frac{1}{\epsilon_o} \rho \quad (22.75)$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_o \vec{j} \quad (22.76)$$

## 22.12 Electromagnetic and Proca Hamiltonian

We'll follow the same steps here as we did for the electromagnetic Lagrangian density. In correspondence with the Hamiltonian densities discussed so far we expect the Hamiltonian density for the electromagnetic four-vector to be expressed by the following.

$$\mathcal{H}_p = \left\{ \begin{array}{l} + \left( -\frac{1}{2} \dot{A}_0^2 - \frac{1}{2} \nabla A_0 \cdot \nabla A_0 + \frac{1}{2} m^2 A_0^2 \right) \\ + \left( -\frac{1}{2} \dot{A}_x^2 - \frac{1}{2} \nabla A_x \cdot \nabla A_x + \frac{1}{2} m^2 A_x^2 \right) \\ + \left( -\frac{1}{2} \dot{A}_y^2 - \frac{1}{2} \nabla A_y \cdot \nabla A_y + \frac{1}{2} m^2 A_y^2 \right) \\ + \left( -\frac{1}{2} \dot{A}_z^2 - \frac{1}{2} \nabla A_z \cdot \nabla A_z + \frac{1}{2} m^2 A_z^2 \right) \end{array} \right. \quad (22.77)$$

Where all four components of  $A^\nu$  have independently the form of the (classical) Hamiltonian field density. These expressions can be written more compact as:

$$\mathcal{H}_p = -\frac{1}{2} \partial^\nu A^\mu \partial^\nu A^\mu + \frac{1}{2} m c^2 A^\mu A^\mu \quad (22.78)$$

We used  $\partial^\nu$  instead of  $\partial_\nu$  here for later convenience. Both give the same end result due to the square. This expression is again not complete. We have to replace the derivatives of  $A^\mu$  in the following sense.

$$\partial^\mu A^\nu \implies \partial^\mu A^\nu - \partial^\nu A^\mu, \quad (22.79)$$

For the same reason as explained in the section on the Lagrangian density. With a zero mass field normalized for SI units this leads us to the well know expression for the energy density of the electromagnetic field.

$$\mathcal{H} = \frac{1}{4\mu_o} \left( \partial^\nu A^\mu - \partial^\mu A^\nu \right) \left( \partial^\nu A^\mu - \partial^\mu A^\nu \right) \quad (22.80)$$

$$\mathcal{H} = \frac{1}{4\mu_o} F^{\mu\nu} F^{\mu\nu} = \frac{1}{2} (\text{BH} + \text{DE}) \quad (22.81)$$